Int. J. Solids Structures Vol. 14, pp. 579–590
© Pergamon Press Ltd., 1978 Printed in Great Britain

0020-7683/78/0701-0579/$02.00/0

FINITE EXTENSION OF A VISCOELASTIC MULTIPLE FILAMENT YARN†

N. C. HUANG
Department of Aerospace and Mechanical Engineering, University of Notre Dame, Notre Dame, IN 46556, U.S.A.

(*Received* 2 *September* 1977; *in revised form* 27 *December* 1977; *received for publication* 15 *February* 1978)

Abstract—This paper is concerned with the investigation of finite extension of a viscoelastic multiple filament and single layered yarn subjected to axial forces and twisting moments. The filament in the yarn is considered as a linear viscoelastic slender curved rod with circular cross section and helical configuration. The theory of slender curved rods is employed in the analysis whereby the curvature of the filament is assumed to be sufficiently small such that the cross section of the filament perpendicular to the axis of the yarn is approximately elliptical. In our study, we have also assumed that there is slipping between filaments during deformation of the yarn. Geometrical nonlinearity in introduced by the resuctions in helical angle and cross section of the filament. In order to illustrate the technique introduced here, in our numerical analysis, the extensional relaxation modulus of the filament is derived from a model of three-element solid and Poisson's ratio of the filament is regarded as constant. Examples are presented for the extensions of yarns with fixed ends and yarns with free ends.

NOTATION

a_0, a	radii of filaments
$\bar{a}$	eqn (51)
c	coefficient of kinetic friction
d	distance from the center of the yarn to the point of tangency of filaments
$\bar{d}$	eqn (51), Fig. 1(b)
E	tensile relaxation modulus
$\bar{E}$	eqn (70)
E_0	$E(0)$
F_λ, F_μ, F_ν	components of the stress resultant vector
F	applied axial force
f_λ, f_ν, f	eqn (51)
h_0	pitch of the helix
$\hat{i}, \hat{j}, \hat{k}$	unit vectors
k_0, k	constants
$\bar{k}_0, \bar{k}$	eqn (51)
L_0, L	lengths of yarns
l_0, l	lengths of filaments
M_λ, M_μ, M_ν	components of the moment vector
M	applied twisting moment
$m_\lambda, m_\mu, m_\nu, \bar{m}$	eqn (51)
n	number of filaments
P	constant force between filaments
p	eqn (51)
r_0, r	radial coordinate of the filament
$\bar{r}_0, \bar{r}$	eqn (51)
T	time
t	T/β
x_1, x_2, x_3	rectangular Cartesian coordinates
β	Fig. 1(c)
Δ, δ	small increments
γ	constant
ϵ_y	axial strain of the yarn
ϵ_λ	axial strain of the filament
θ_0, θ	helical angles
κ_0, κ	principal normal curvatures
$\hat{\lambda}_0, \hat{\mu}_0, \hat{\nu}_0$	unit vectors
$\bar{\rho}_0, \bar{\rho}$	eqns (6) and (11)
σ	Poisson's ratio
τ_0, τ	torsions
ϕ	polar angle
Ω	angle of twist of the yarn per unit length
ω	eqn (51)

†This research is supported by the National Science Foundation Grant ENG76-05775.

INTRODUCTION

The mechanical behavior of most textile materials exhibits what is known as time-sensitivity. For a yarn subjected to an axial load, creep deformation is observed following instantaneous elastic response. In order to analyze the yarn problem in a more realistic manner, the theory of viscoelasticity must be employed. The creep deformation of linearly viscoelastic continuous filament yarn has been investigated by Jones[1]. His analysis is based on the small deformation theory whereby geometrical nonlinearity is ignored. Also in Jones' study, the yarn material is considered to be incompressible; the normal stresses acting on any yarn element in the transverse directions are assumed to be equal and the shearing stresses are neglected. The finite extension of a linearly viscoelastic two-ply filament yarn has been analyzed by Huang[2] based on the theory of slender curved rods[3]. In Huang's study, geometrical nonlinearity is introduced as a result of reductions in helical angle and filament cross section.

In this paper, we shall study the problem of finite extension of multiple filament linearly viscoelastic yarns subjected to axial forces and twisting moments at both ends. Similar to the case of the two-ply filament yarn, we shall treat the filament as a slender curved rod with circular cross section and helical configuration. We shall assume that the curvature of the filament is sufficiently small such that the cross section of the filament perpendicular to the yarn axis is approximately elliptical. Also, we shall consider that there is slipping between filaments during deformation of the yarn. It is found that the creep deformation of the yarn is governed by two nonlinear integral equations which is solved numerically by a modified Newton's method. Two problems have been selected for study, namely, the extension of a yarn with fixed ends and the extension of a yarn with free ends. Effects of initial helical angle and the superposition of a twisting moment on the axial extension of the yarn have also been included in the study.

GEOMETRY OF THE YARN

Let us consider a long linearly viscoelastic n-ply filament yarn as shown in Fig. 1(a). ($n = 2, 3, 4, \ldots$). The yarn consists of a single layer of filaments which form a cylindrical tube. The cross section of the filament in the undeformed state is considered to be circular with radius a_0. The center line of the undeformed filament is prescribed by the following rectangular Cartesian coordinates:

$$x_1 = r_0 \cos \phi_0, \quad x_2 = r_0 \sin \phi_0, \quad x_3 = k_0 \phi_0, \tag{1}$$

where r_0 and ϕ_0 are the polar coordinates and k_0 is a constant which is related to the length of the filament of one turn of twist measured along the axis of the yarn h_0 by

$$h_0 = 2\pi k_0.$$

The unit vectors in the tangential, principal normal and binormal directions of the center line of the undeformed filament are

$$\hat{\lambda}_0 = \frac{1}{\rho_0}(-r_0 \sin \phi_0 \hat{\imath} + r_0 \cos \phi_0 \hat{\jmath} + k_0 \hat{k}), \tag{3}$$

$$\hat{\mu}_0 = -(\cos \phi_0 \hat{\imath} + \sin \phi_0 \hat{\jmath}), \tag{4}$$

$$\hat{\nu}_0 = \frac{1}{\rho_0}(k_0 \sin \phi_0 \hat{\imath} - k_0 \cos \phi_0 \hat{\jmath} + r_0 \hat{k}) \tag{5}$$

respectively, where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are unit vectors in the x_1, x_2 and x_3-directions and

$$\rho_0 = (r_0^2 + k_0^2)^{1/2}. \tag{6}$$

Note that the principal normal of the filament is in the radial direction and toward the axis of the yarn. The helical angle of the center line of the undeformed filament is

$$\theta_0 = \tan^{-1} \frac{r_0}{k_0}. \tag{7}$$

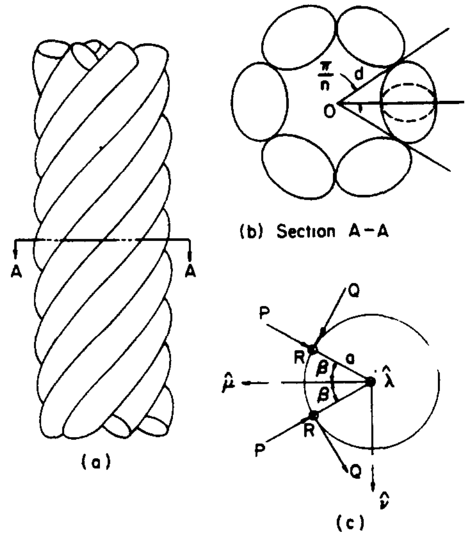

Fig. 1. Geometry of the problem.

Let us define the curvature vector by $d\mathbf{H}/ds$ where $d\mathbf{H}$ is the infinitesimal rotation vector of the coordinate axes in $\hat{\lambda}$, $\hat{\mu}$ and $\hat{\nu}$-directions in a distance ds along the center line of the filament. The principal normal curvature of the center line of the undeformed filament is then the component of the curvature in the binormal direction. The torsion of the center line is the component of the curvature in the tangential direction.† They are

$$\kappa_0 = \frac{1}{\rho_0}\sin\theta_0, \qquad \tau_0 = \frac{1}{\rho_0}\cos\theta_0 \tag{8}$$

respectively. The component of the curvature in the principal normal direction is zero.

Let us cut the yarn by a plane perpendicular to the axis of the yarn. The cross section is shown in Fig. 1(b). We shall follow Phillips and Costello[5] and assume that the curvature of the filament is sufficiently small such that the cross section of each filament is elliptical. Hence the cross section of the undeformed yarn consists of n ellipses tangent to each other with center of ellipse at $(r_0\cos(2i\pi/n),\ r_0\sin(2i\pi/n))$ $(i = 0, 1, \ldots n-1)$. We shall also assume that the common tangent of any two neighboring ellipses passes through the center of the cross section. Since the principal normal of the filament is in the radial direction, the major axis of the ellipse is $2a_0\rho_0/k_0$ and the minor axis is $2a_0$. By analytical geometry, we can easily show that

$$r_0 = a_0k_0\left(k_0^2\sin^2\frac{\pi}{n} - a_0^2\cos^2\frac{\pi}{n}\right)^{-1/2}. \tag{9}$$

Equation (9) agrees with the corresponding relation given in [4]. When $n = 2$, we obtain from equation (9) that $r_0 = a_0$ which agrees with the result of [2].

The yarn is subjected to an axial tension $\bar{F}$ and a twisting moment $\bar{M}$ in the direction of the original twist of the yarn. In the deformed state, as a result of contact deformation, the cross section of the filament is no longer circular. Since the configuration of filament is helical, the analysis of this type of contact problem is extremely difficult. In the following, we shall neglect the contact deformation and assume that the cross section of the filament in the deformed state remains circular with radius a. The center line of the filament in the deformed state remains helical. Let r and ϕ be the polar coordinates of any point on the deformed center line of the filament. We have equivalent equations identical to eqns (1)–(9) with the subscript 0 deleted. Hence,

$$\kappa = \frac{1}{\rho}\sin\theta, \qquad \tau = \frac{1}{\rho}\cos\theta, \tag{10}$$

†In Love's text[4], our torsion is referred to as the twist of the filament.

where

$$\rho = (r^2 + k^2)^{1/2}, \qquad \theta = \tan^{-1}\frac{r}{k}. \tag{11}$$

Also, we have

$$r = ak\left(k^2 \sin^2\frac{\pi}{n} - a^2 \cos^2\frac{\pi}{n}\right)^{-1/2}. \tag{12}$$

The distance from the center of the cross section of the yarn to the point of tangency of two neighboring ellipses is found as

$$d = \frac{a}{k}(a^2 + k^2)\left(k^2 \sin^2\frac{\pi}{n} - a^2 \cos^2\frac{\pi}{n}\right)^{-1/2} \cos\frac{\pi}{n}. \tag{13}$$

The line of contact of the deformed filament is helical with radius d.

Next, let us consider a cross section of the deformed filament as shown in Fig. 1(c). To determine the central angle 2β between the points of contact, let us first project the circular cross section on the plane perpendicular to the axis of the yarn. The projection is also an ellipse with major axis $2a$ and minor axis $2ak/\rho$ as shown by dotted line in Fig. 1(b). Note that the projection of the helical line of contact on the plane perpendicular to the axis of the yarn is a circle with radius d and center at 0. The angle β can be determined from the intersection of the projection of the cross section and the projection of the line of contact. Similar approach for the determination of β has also been employed by Costello and Phillips[6]. It is found that

$$\cos\beta = \{\rho^2 - [\rho^2(d^2 + k^2) - a^2k^2]^{1/2}\}/(ar). \tag{14}$$

Thus after d is calculated by eqn (13), β can be determined by eqn (14). Note that for the two-ply filament yarns, $n = 2$, $d = \beta = 0$. Hence the filaments are in contact to each other at the center line of the yearn. The same conclusion has also been drawn in [2].

FORMULATION OF THE PROBLEM

In the following, we shall denote the components of any quantity in the tangential, principal normal and binormal directions by subscripts λ, μ and ν respectively. Each filament will be treated as a long slender curved rod. The components of the stress resultant acting on the cross section of the filament are denoted by F_λ, F_μ and F_ν and the components of moment acting on the cross section are denoted by M_λ, M_μ and M_ν. The components of the distributed force per unit length of the filament are p_λ, p_μ and p_ν and the components of the distribution moment per unit length are m_λ, m_μ and m_ν.

Along the line of contact, there exist the normal distributed force P and tangential distributed forces Q and R as shown in Fig. 1(c). Hence

$$p_\lambda = 0, \quad p_\mu = -2P\cos\beta, \quad p_\nu = 2Q\cos\beta \tag{15}$$

and

$$m_\lambda = 2Qa, \quad m_\mu = 2Ra\sin\beta, \quad m_\nu = 0. \tag{16}$$

If we treat the filament as a one-dimensional slender body, we can obtain the equations of equilibrium of all forces and moments acting on an element of the filament from the theory of slender curved rods[3]. Since the yarn is considered to be long, all derivatives of the stress results and moments with respect to the arc length of the filament must vanish. The equations of equilibrium can be written as

$$F_\mu\kappa - p_\lambda = 0, \tag{17}$$

$$F_\lambda \kappa - F_\nu \tau + p_\mu = 0, \tag{18}$$

$$F_\mu \tau + p_\nu = 0, \tag{19}$$

$$M_\mu \kappa - m_\lambda = 0, \tag{20}$$

$$M_\lambda \kappa - M_\nu \tau - F_\nu + m_\mu = 0, \tag{21}$$

$$M_\mu \tau + F_\mu + m_\nu = 0. \tag{22}$$

With eqns (15) and (16), eqns (17)–(22) become

$$F_\mu = M_\mu = p_\nu = m_\lambda = Q = 0, \tag{23}$$

$$F_\lambda \kappa - F_\nu \tau - 2P \cos \beta = 0, \tag{24}$$

$$M_\lambda \kappa - M_\nu \tau + 2Ra \sin \beta - F_\nu = 0. \tag{25}$$

In the following, we shall consider that during deformation of the yarn, there is slipping between filaments. Let us denote the coefficient of kinetic friction between filaments by c. We have

$$R = cP. \tag{26}$$

Here eqn (25) can be rewritten as

$$M_\lambda \kappa - M_\nu \tau + 2cPa \sin \beta - F_\nu = 0. \tag{27}$$

Let us set the local coordinates in the tangential, principal normal and binormal directions. Under the assumption of small strains, the Clebsch–Basset moment-curvature relations for elastic slender curved rods[4] are

$$M_\lambda = GJ(\tau - \tau_0) = \frac{\pi a^4}{4(1+\sigma)} E(\tau - \tau_0), \tag{28}$$

$$M_\mu = 0, \tag{29}$$

$$M_\nu = EI(\kappa - \kappa_0) = \frac{\pi a^4}{4} E(\kappa - \kappa_0), \tag{30}$$

where EI is the bending stiffness, GJ is the torsional stiffness and σ is Poisson's ratio. Equation (29) is derived based on the fact that the component of curvature in the principal normal direction is zero for both the undeformed and deformed filaments. In the visco-elastic case, in view of the absence of data, we shall assume Poisson's ratio to be constant and treat Young's modulus E in eqns (28) and (30) as an integral operation according to the correspondence principle in viscoelasticity. In the following, we shall use the notation

$$F^* \mathrm{d}f = \int_{-\infty}^{T} F(T - t_1) \frac{\mathrm{d}f(t_1)}{\mathrm{d}t_1} \mathrm{d}t_1, \tag{31}$$

where T is the time. In the viscoelastic case eqns (28) and (30) are

$$M_\lambda = \frac{\pi a^4}{4(1+\sigma)} E^* d(\tau - \tau_0) = \frac{\pi a^4}{4(1+\sigma)} E^* d\left(\frac{k}{\rho^2}\right), \tag{32}$$

$$M_\nu = \frac{\pi a^4}{4} E^* d(\kappa - \kappa_0) = \frac{\pi a^4}{4} E^* d\left(\frac{r}{\rho^2}\right), \tag{33}$$

where $E(T)$ is the axial tensile relaxation modulus of the filament. Equation (29) still holds. By eqns (10), (24), (27), (32) and (33), we obtain

$$F_\nu = a(\rho^2 + cak \tan\beta)^{-1}\left\{\frac{\pi a^3}{4}\left[\frac{r}{1+\sigma}E^*d\left(\frac{k}{\rho^2}\right) - kE^*d\left(\frac{r}{\rho^2}\right)\right] + crF_\lambda \tan\beta\right\}. \tag{34}$$

The contact force between filament can be found from eqn (24) as

$$P = \frac{1}{2\cos\beta}(F_\lambda\kappa - F_\nu\tau). \tag{35}$$

In the elastic case, as a result of Poisson's effect, the normal strain in the transverse direction of the filament is

$$\frac{a}{a_0} - 1 = -\sigma\frac{F_\lambda}{\pi a^2 E}. \tag{36}$$

By means of the correspondence principle, the tangential component of stress resultant in the filament in the viscoelastic case is

$$F_\lambda = -\frac{\pi a^2}{\sigma a_0}E^*da. \tag{37}$$

The overall equilibrium of the internal forces and the applied force $\bar{F}$ requires that [3, 5].

$$\bar{F} = n(F_\lambda\cos\theta + F_\nu\sin\theta) = \frac{n}{\rho}(kF_\lambda + rF_\nu). \tag{38}$$

Similarly, the overall equilibrium in moments requires

$$\bar{M} = \frac{n}{\rho}(kM_\lambda + rM_\nu + r^2F_\lambda - rkF_\nu). \tag{39}$$

With eqns (32) and (33), we can rewrite eqn (39) as

$$\bar{M} = \frac{n}{\rho}\left\{\frac{\pi a^4}{4}\left[\frac{k}{1+\sigma}E^*d\left(\frac{k}{\rho^2}\right) + rE^*d\left(\frac{r}{\rho^2}\right)\right] + r(rF_\lambda - kF_\nu)\right\}. \tag{40}$$

Let us consider a yarn of M_0 turns in the undeformed state. After deformation it changes to M turns. Hence the undeformed and deformed lengths of the yarn measured in the direction of yarn axis are

$$L_0 = 2\pi M_0k_0, \qquad L = 2\pi Mk \tag{41}$$

respectively. The undeformed and deformed length of the filament are

$$l_0 = 2\pi M_0\rho_0, \qquad l = 2\pi M\rho \tag{42}$$

respectively. Hence the axial strain of the yarn is

$$\epsilon_y = \frac{L}{L_0} - 1 = \frac{Mk}{M_0k_0} - 1 \tag{43}$$

and the axial strain of the filament is

$$\epsilon_\lambda = \frac{l}{l_0} - 1 = \frac{M\rho}{M_0\rho_0} - 1. \tag{44}$$

As a result of Poisson's effect, we have

$$\frac{a}{a_0}-1=-\sigma\epsilon_\lambda. \tag{45}$$

After elimination of ϵ_λ and M/M_0 from eqns (43)–(45), we obtain

$$\epsilon_y=\frac{k\rho_0}{k_0\rho}\left[1+\frac{1}{\sigma}\left(1-\frac{a}{a_0}\right)\right]-1. \tag{46}$$

The angle of twist of the yarn per unit length is

$$\Omega=\frac{(M-M_0)2\pi}{L_0}=\frac{1}{k_0}\left(\frac{M}{M_0}-1\right). \tag{47}$$

After elimination of ϵ_λ and M/M_0 from eqns (44), (45) and (47), we obtain

$$\Omega=\frac{1}{k_0}\left\{\frac{\rho_0}{\rho}\left[1+\frac{1}{\sigma}\left(1-\frac{a}{a_0}\right)\right]-1\right\}. \tag{48}$$

Hence if the ends of the yarn are unconstrained, elongation of the yarn will be coupled by a twist. On the other hand, if the ends of the yarn are constrained such that $\Omega=0$, we have an additional constrained condition from eqn (48). It is

$$\rho=\rho_0\left[1+\frac{1}{\sigma}\left(1-\frac{a}{a_0}\right)\right]. \tag{49}$$

In this case, if a is known, ρ can be determined from eqn (49) and k can be found from the relation derived from eqns (11) and (12).

$$k^2=\frac{1}{2}\left\{\rho^2-a^2+\left[(\rho^2-a^2)^2-4a^2\rho^2\cot^2\frac{\pi}{n}\right]^{1/2}\right\}. \tag{50}$$

Finally, θ can be found from eqn (11).

METHOD OF SOLUTION

Let us denote $E(0)$ by E_0 and introduce the following dimensionless quantities:

$$\bar{E}(T)=E(T)/E_0,\quad \bar{k}_0=k_0/a_0,\quad \bar{r}_0=r_0/a_0,\quad \bar{\rho}_0=\rho_0/a_0,\quad \bar{a}=a/a_0,\quad \bar{k}=k/a_0,$$
$$\bar{r}=r/a_0,\quad \bar{\rho}=\rho/a_0,\quad \bar{d}=d/a_0,\quad f_\lambda=\frac{F_\lambda}{E_0\pi a_0^2},\quad f_\nu=\frac{F_\nu}{E_0\pi a_0^2},$$
$$p=\frac{P}{E_0\pi a_0},\quad \bar{f}=\frac{\bar{F}}{E_0\pi a_0^2},\quad \bar{m}=\frac{\bar{M}}{E_0\pi a_0^3},\quad \omega=\Omega a_0. \tag{51}$$

For given values of n, θ_0, the initial geometry of the yarn is determined from the following equations:

$$\bar{k}_0=\csc\frac{\pi}{n}\left(\cot^2\theta_0+\cos^2\frac{\pi}{n}\right)^{1/2}, \tag{52}$$

$$\bar{r}_0=\bar{k}_0\tan\theta_0, \tag{53}$$

$$\bar{\rho}_0=(\bar{r}_0^2+\bar{k}_0^2)^{1/2}. \tag{54}$$

Under given values of c, σ, $\bar{f}$ and $\bar{m}$ and given function $\bar{E}(T)$, the viscoelastic deformation of the yarn is governed by the following equations:

$$\bar{r} = \bar{a}\bar{k}\left(\bar{k}^2 \sin^2\frac{\pi}{n} - \bar{a}^2\cos^2\frac{\pi}{n}\right)^{-1/2}, \tag{55}$$

$$\bar{\rho} = (\bar{r}^2 + \bar{k}^2)^{1/2}, \tag{56}$$

$$\bar{d} = \frac{\bar{a}}{\bar{k}}(\bar{a}^2 + \bar{k}^2)\left(\bar{k}^2 \sin^2\frac{\pi}{n} - \bar{a}^2\cos^2\frac{\pi}{n}\right)^{-1/2}\cos\frac{\pi}{n}, \tag{57}$$

$$\cos\beta = \{\bar{\rho}^2 - [\bar{\rho}^2(\bar{d}^2 + \bar{k}^2) - \bar{a}^2\bar{k}^2]^{1/2}\}/(\bar{a}\bar{r}), \tag{58}$$

$$f_\lambda = -\frac{1}{\sigma}\bar{a}^2\bar{E}*d\bar{a}, \tag{59}$$

$$f_\nu = \bar{a}(\bar{\rho}^2 + c\bar{a}\bar{k}\tan\beta)^{-1}\left\{\frac{\bar{a}^3}{4}\left[\frac{\bar{r}}{1+\sigma}\bar{E}*d\left(\frac{\bar{k}}{\bar{\rho}^2}\right) - \bar{k}\bar{E}*d\left(\frac{\bar{r}}{\bar{\rho}^2}\right)\right] + c\bar{r}f_\lambda\tan\beta\right\}, \tag{60}$$

$$p = (\bar{r}f_\lambda - \bar{k}f_\nu)/(2\bar{\rho}^2\cos\beta), \tag{61}$$

$$g = \bar{f} - \frac{n}{\bar{\rho}}(\bar{k}f_\lambda + \bar{r}f_\nu) = 0, \tag{62}$$

$$h = \bar{m} - \frac{n}{\bar{\rho}}\left\{\frac{\bar{a}^4}{4}\left[\frac{\bar{k}}{1+\sigma}\bar{E}*d\left(\frac{\bar{k}}{\bar{\rho}^2}\right) + \bar{r}\bar{E}*d\left(\frac{\bar{r}}{\bar{\rho}^2}\right)\right] + \bar{r}(\bar{r}f_\lambda - \bar{k}f_\nu)\right\} = 0, \tag{63}$$

$$\epsilon_y = \frac{\bar{k}\bar{\rho}_0}{\bar{k}_0\bar{\rho}}\left[1 + \frac{1}{\sigma}(1-\bar{a})\right] - 1, \tag{64}$$

$$\omega = \frac{1}{\bar{k}_0}\left\{\frac{\bar{\rho}_0}{\bar{\rho}}\left[1 + \frac{1}{\sigma}(1-\bar{a})\right] - 1\right\}. \tag{65}$$

Equations (62) and (63) are simultaneous equations for the calculation of $\bar{a}(T)$ and $\bar{k}(T)$. The external force $\bar{F}$ and moment $\bar{M}$ are applied at $T = 0$. The instantaneous response at $T = 0^+$ is elastic. The governing equations for the instantaneous elastic response are

$$f_\lambda = \frac{1}{\sigma}\bar{a}^2(1-\bar{a}), \tag{66}$$

$$f_\nu = \bar{a}(\bar{\rho}^2 + c\bar{a}\bar{k}\tan\beta)^{-1}\left\{\frac{\bar{a}^3}{4}\left[\frac{\bar{r}}{1+\sigma}\left(\frac{\bar{k}}{\bar{\rho}^2} - \frac{\bar{k}_0}{\bar{\rho}_0^2}\right) - \bar{k}\left(\frac{\bar{r}}{\bar{\rho}^2} - \frac{\bar{r}_0}{\bar{\rho}_0^2}\right)\right] + c\bar{r}f_\lambda\tan\beta\right\}, \tag{67}$$

$$g = \bar{f} - \frac{n}{\bar{\rho}}(\bar{k}f_\lambda + \bar{r}f_\nu) = 0, \tag{68}$$

$$h = \bar{m} - \frac{n}{\bar{\rho}}\left\{\frac{\bar{a}^4}{4}\left[\frac{\bar{k}}{1+\sigma}\left(\frac{\bar{k}}{\bar{\rho}^2} - \frac{\bar{k}_0}{\bar{\rho}_0^2}\right) + \bar{r}\left(\frac{\bar{r}}{\bar{\rho}^2} - \frac{\bar{r}_0}{\bar{\rho}_0^2}\right)\right] + \bar{r}(\bar{r}f_\lambda - \bar{k}f_\nu)\right\} = 0. \tag{69}$$

In order to illustrate our method of analysis, we shall consider that the viscoelastic characteristics of the filament can be derived from a model of three-element solid which consists of a spring and a Kelvin element (i.e. a spring and dashpot parallel) in series. The relaxation modulus of the three-element solid as expressed in a dimensionless form can be found in [7]. It is [7]. It is

$$\bar{E}(T) = \gamma + (1-\gamma)\,e^{-T/\beta}, \tag{70}$$

where β and γ are material constants. The dimensionless delayed elastic modulus γ has its value in the range $0 \le \gamma < 1$. Let us define a dimensionless time $t = T/\beta$. Equation (70) becomes

$$\bar{E}(t) = \gamma + (1-\gamma)\,e^{-t}. \tag{71}$$

In our computation, we shall employ Lee and Roger's method[8] to evaluate the hereditary integral numerically. In our numerical scheme, we first divide t into N intervals with $t_1 = 0^+$ and $t_{N+1} = T$. We have

$$\bar{E}^* df = S_{N+1} + \frac{1}{2}[1 + \bar{E}(t_{N+1} - t_N)]f(t_{N+1}), \tag{72}$$

where

$$S_{N+1} = E(t_{N+1})[f(t_1) - f_0] - \frac{1}{2}[1 + E(t_{N+1} - t_N)]f(t_N)$$
$$+ \frac{1}{2}\sum_{i=1}^{N-1}\{[\bar{E}(t_{N+1} - t_{i+1}) + \bar{E}(t_{N+1} - t_i)][f(t_{i+1}) - f(t_i)]\} \tag{73}$$

and f_0 is the value of f in the undeformed state. Note that when time increases step by step, at $t = t_{N+1}$, S_{N+1} involves $f(t_i)$ for $i = 1, 2, \ldots N$ which have been determined in the previous time steps. The only unknown in eqn (73) is $f(t_{N+1})$. By using eqn (73), an integral equation of Volterra's type can be reduced to an algebraic equation.

In the following, we shall analyze the problem of finite extension of yarns with two types of end conditions.

Yarn with fixed ends

When the ends of the yarn are fixed, $\omega = 0$. The fixed-end twisting moment is determined by eqn (69) for the instantaneous elastic response and by eqn (63) for the viscoelastic response. For each time step, we assume a value for $\bar{a}$ and compute $\bar{\rho}$ and $\bar{k}$ by

$$\bar{\rho} = \bar{\rho}_0\left[1 + \frac{1}{\sigma}(1 - \bar{a})\right] \tag{74}$$

and

$$\bar{k}^2 = \frac{1}{2}\left\{\bar{\rho}^2 - \bar{a}^2 + \left[(\bar{\rho}^2 - \bar{a}^2)^2 - 4\bar{a}^2\bar{\rho}^2\cot^2\frac{\pi}{n}\right]^{1/2}\right\}. \tag{75}$$

The values of $\bar{r}$, $\bar{d}$, β, f_λ and f_ν are determined respectively by eqns (55), (57)–(60). The correct value of $\bar{a}$ must fulfill the condition that the value of g as determined by eqn (68) or (62) is zero. We shall use a modified Newton's iterative method for the determination of $\bar{a}$. In this procedure we select three values of $\bar{a}$ as given by $\bar{a}_i$, $\bar{a}_i - \Delta$ and $\bar{a}_i + \Delta$, where Δ is a small number, and compute values of g which are denoted by g_1, g_2 and g_3. The derivative $dg/d\bar{a}$ at $\bar{a} = \bar{a}_i$ can be approximated by the following central difference equations:

$$g' = \left.\frac{dg}{d\bar{a}}\right|_{\bar{a}=\bar{a}_i} = \frac{1}{2\Delta}(g_3 - g_2). \tag{76}$$

Hence, according to Newton's iterative formula, the new value of $\bar{a}$ in the iteration would be

$$\bar{a}_{i+1} = \bar{a}_i - \frac{g_1}{g'} = \bar{a}_i - \frac{2\Delta g_1}{g_3 - g_2}. \tag{77}$$

Our iterative procedure continues until the absolute value of g is smaller than a certain prescribed value. It is noted that in our problem, the rate of convergence is fast.

Computations are first carried out for $\theta_0 = 15°$, $n = 4$, $\gamma = 0.7$, $c = 0.2$ and $\sigma = 0.45$. In Fig. 2, the creep curves $\epsilon_y(t)$ are shown by solid lines for various values of $\bar{f}$. The finite jumps at $t = 0$ indicate the instantaneous elastic response of the yarn. At $t = \infty$, all creep curves approach asymptotically the value corresponding to the delayed elastic modulus $\bar{E}(\infty) = 0.7$. The fixed end twisting moments $\bar{m}(t)$ are also shown by dotted lines. It is noted that the fixed end twisting moment is nearly constant for this case. In order to show the geometrical nonlinearity, $\epsilon_y(1)$ is plotted against $\bar{f}$ in Fig. 3. The $\epsilon_y(1)$ curve as determined by the linear theory is also shown by a dotted line for comparison. Note that the nonlinearity becomes evident when $\bar{f}$ is large. The helical angle of the filament $\theta(1)$ is plotted against $\bar{f}$ in Fig. 3. As we may expect, $\theta(1)$ decreases with increasing $\bar{f}$. In Fig. 4, the dependence of $\epsilon_y(1)$, $f_\lambda(1)$ and $\bar{m}(1)$ on n are shown for $\bar{f} = 0.02$. When n increases, both $\epsilon_y(1)$ and $f_\lambda(1)$ decrease. The value of $\bar{m}(1)$ increases with n because the radial distance $\bar{r}(1)$ increases with n. The dependence of $f_\lambda(1)$ and $\bar{m}(1)$ on the initial helical angle θ_0 is shown in Fig. 5 for $n = 4$ and $\bar{f} = 0.02$. When $\theta_0 = 0$, all filaments are straight. Hence, $f_\lambda(1) = 0.005$ and $\bar{m}(1) = 0$. It is found that both $f_\lambda(1)$ and $\bar{m}(1)$ increase with θ_0.

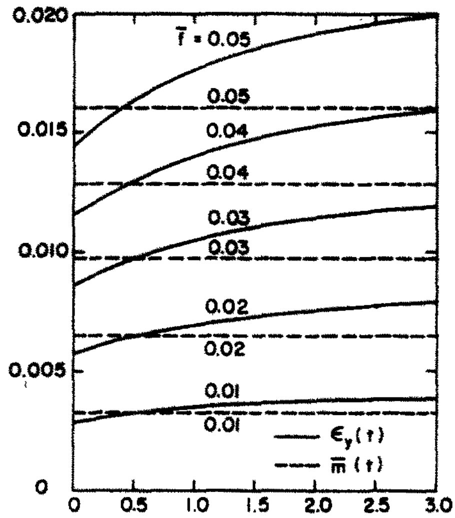

Fig. 2. $\epsilon_y(t)$ and $\bar{m}(t)$ curves for $\theta^0 = 15°$, $\omega = 0$, (fixed ends) $n = 4$, $\gamma = 0.7$, $c = 0.2$, $\sigma = 0.45$ and various values of $\bar{f}$.

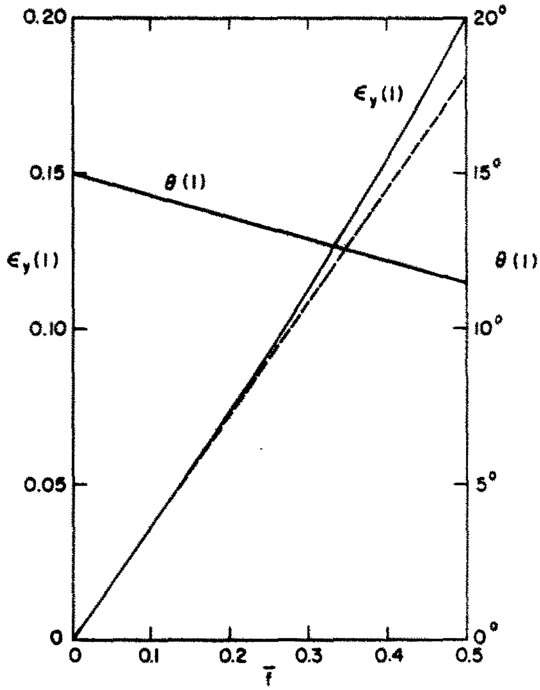

Fig. 3. $\epsilon_y(1)$ and $\theta(1)$ vs $\bar{f}$ curves for $\theta_0 = 15°$, $\omega = 0$, (fixed ends) $n = 4$, $\gamma = 0.7$, $c = 0.2$ and $\sigma = 0.45$.

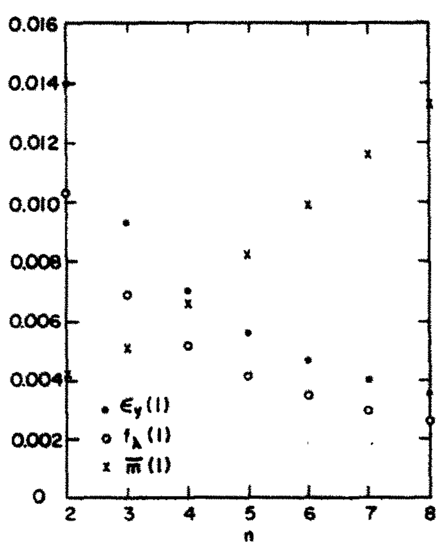

Fig. 4. Values of $\epsilon_y(1)$ and $f_\lambda(1)$ for $\theta_0 = 15°$, $\omega = 0$, (fixed ends) $\gamma = 0.7$, $c = 0.2$, $\sigma = 0.45$, $\bar{f} = 0.02$ and various values of n.

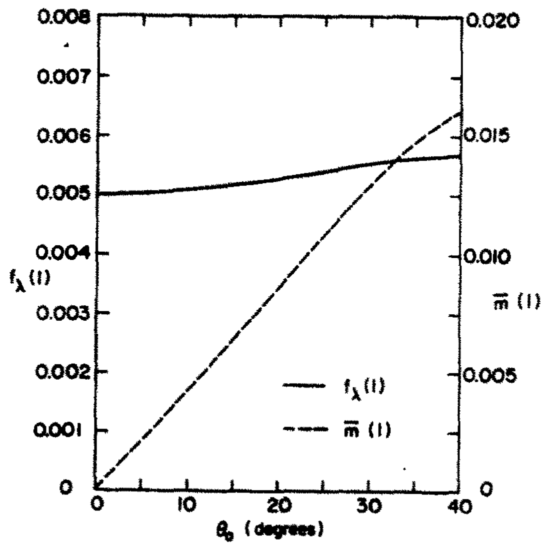

Fig. 5. $f_\lambda(1)$ and $\bar{m}(1)$ curves for $\omega = 0$, (fixed ends), $n = 4$, $\gamma = 0.7$, $c = 0.2$, $\sigma = 0.45$, $\bar{f} = 0.02$ and various values of θ_0.

Yarn with free ends

When the ends of the yarn are free, $\bar{m} = 0$ and $\omega \neq 0$. In this case, we have two unknowns $\bar{a}$ and $\bar{k}$ to be determined. Again, we shall employ a modified Newton's iterative method in our computation. First, we try out a set of values $\bar{a} = \bar{a}_i$ and $\bar{k} = \bar{k}_i$ and compute g and h according to eqns (68) and (69) for the instantaneous elastic response and eqns (62) and (63) for the viscoelastic response. They are denoted by (g_1, h_1). Next, we select four sets of values for $\bar{a}$ and $\bar{k}$ in the neighborhood of $\bar{a} = \bar{a}_i$ and $\bar{k} = \bar{k}_i$. They are $(\bar{a}_i - \Delta, \bar{k}_i)$, $(\bar{a}_i + \Delta, \bar{k}_i)$, $(\bar{a}_i, \bar{k}_i - \delta)$, $(\bar{a}_i, \bar{k}_i + \delta)$. We have four additional sets of calculated values of g and h. They are (g_2, h_2), (g_3, h_3), (g_4, h_4) and (g_5, h_5) respectively. The partial derivatives of g and h with respect to $\bar{a}$ and $\bar{k}$ as evaluated at $\bar{a} = \bar{a}_i$ and $\bar{k} = \bar{k}_i$ can be approximated by the following central difference equations:

$$\frac{\partial g}{\partial \bar{a}} = \frac{1}{2\Delta}(g_3 - g_2), \quad \frac{\partial h}{\partial \bar{a}} = \frac{1}{2\Delta}(h_3 - h_2), \tag{78}$$

$$\frac{\partial g}{\partial \bar{k}} = \frac{1}{2\delta}(g_5 - g_4), \quad \frac{\partial h}{\partial \bar{k}} = \frac{1}{2\delta}(h_5 - h_4). \tag{79}$$

The new set of $\bar{a}$ and $\bar{k}$ can be determined by the following equations given in Newton's iterative method for two variables:

$$\bar{a}_{i+1} = \bar{a}_i - \frac{1}{D}\left(g_1 \frac{\partial h}{\partial \bar{k}} - h_1 \frac{\partial g}{\partial \bar{k}}\right), \quad \bar{k}_{i+1} = \bar{k}_i - \frac{1}{D}\left(h_1 \frac{\partial g}{\partial \bar{a}} - g_1 \frac{\partial h}{\partial \bar{a}}\right), \tag{80}$$

where

$$D = \frac{\partial g}{\partial \bar{a}}\frac{\partial h}{\partial \bar{k}} - \frac{\partial g}{\partial \bar{k}}\frac{\partial h}{\partial \bar{a}}. \tag{81}$$

The iterative procedure continues until the value of $\bar{a}^2 + \bar{k}^2$ is smaller than a certain prescribed value. It is found that the rate of convergence is also fast for the case of two variables.

The creep curves, $\epsilon_y(t)$, are shown by solid lines in Fig. 6 for $\theta_0 = 15°$, $n = 4$, $\gamma = 0.7$, $c = 0.2$, $\sigma = 0.45$ and different values of $\bar{f}$. In comparison with the creep curves shown in Fig. 2, it is seen that additional axial strain of the yarn can be introduced if the ends of the yarn is allowed to twist. It is found that when the ends of the yarn is free, the axial extension of the yarn is accompanied by an untwist of the yarn. The angles of untwist per unit length of the yarn, $-\omega(t)$, are also shown in Fig. 6 by dotted lines. The effect of superposition of a twisting moment on the elongation of the yarn is shown in Fig. 7. It is noted that when $\bar{m}$ increases, the helical angle increases and consequently $\epsilon_y(1)$ decreases. When $\bar{m}$ is sufficiently large, $\epsilon_y(1)$ can become negative. The filament stress $f_\lambda(1)$ and the contact force $p(1)$ are also plotted against $\bar{m}$ in Fig. 7. It is found that $f_\lambda(1)$ is essentially governed by $\bar{f}$ and the effect of the additional twisting moment on $f_\lambda(1)$ is insignificant. The contact force $p(1)$ increases slightly with $\bar{m}$.

DISCUSSION

(1) The finite extension of elastic wire cables has been investigated by Costello and Phillips[9]. Their problem corresponds to our problem with $T = 0$. The conclusions drawn in their study are very much similar to what we have obtained here for the viscoelastic problem.

(2) The mechanical behavior of a yarn is governed chiefly by three factors—time, temperature and humidity. In this paper, our investigation is focused on the time-dependent characteristics of the yarn and ignored the effects of temperature and humidity. Hence our problem is restricted to the case of isothermal environment with constant humidity. Should the temperature and humidity also be time dependent the relaxation modulus used in our analysis must be modified to include the effect of temperature and humidity.

(3) Our analysis is based on a model where the deformed cross section of the filament remains circular. In reality, the contract of filaments are of Hertz's type in both undeformed

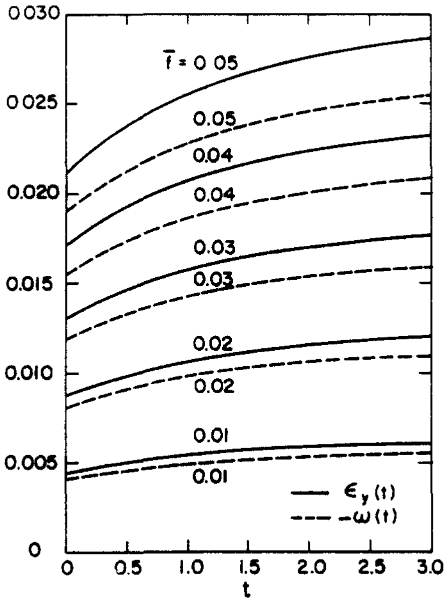

Fig. 6. $\epsilon_y(t)$ and $-\omega(t)$ curves for $\theta_0 = 15°$, $\bar{m} = 0$, (free ends), $n = 4$, $\gamma = 0.7$, $c = 0.2$, $\sigma = 0.45$ and various values of $\bar{f}$.

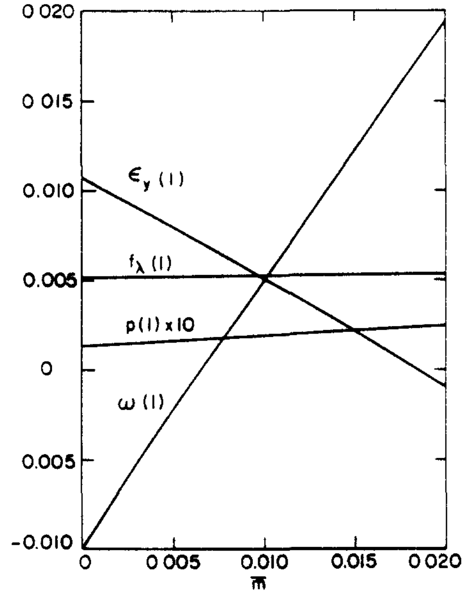

Fig. 7. $\epsilon_y(1)$, $f_\lambda(1)$, $\omega(1)$ and $p(1) \times 10$ vs $\bar{m}$ curves for the case of free ends with $\theta_0 = 15°$, $f = 0.02$, $n = 4$, $\gamma = 0.7$, $c = 0.2$ and $\sigma = 0.45$.

and deformed states. Additional geometrical nonlinearity can be introduced by the Hertz's contact deformation of the filament. The result given in this paper is valid if the width of contact is small in comparison with the diameter of the filament.

(4) It is noted that the actual cross sections of filaments are not perfectly circular. Also their distribution is not entirely uniform. To analyze a yarn with nonuniform filaments, statistical theory must be employed. However, our study would still provide a general feature of the finite deformation of viscoelastic multiple filament yarns.

REFERENCES

1. N. Jones, Elastic–plastic and viscoelastic behavior of a continuous filament yarn. *Int. J. Mech. Sci.* **16**, 679–687 (1974).
2. N. C. Huang, On finite extension of a viscoelastic two-ply filament yarn. *Textile Res. J.* **48**, 61–67 (1978).
3. N. C. Huang, Theories of elastic slender curved rods, *J. Appl. Math. Phys.* (*ZAMP*), **24** 1–19 (1973).
4. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Chap. 18, p. 397. Dover, New York.
5. J. W. Phillips and G. A. Costello, Contact stress in twisted wire cables. *J. Engng Mech. Div. Proc. American Soc. Civil Engrs.* **99**, EM2, 331–341 (1973).
6. G. A. Costello and J. W. Phillips, A more exact theory of twisted wire cables. *J. Engng Mech. Div. Proc. American Soc. Civil Engrs*, **100**, EM5, 1096–1099 (1974).
7. W. Flügge, *Viscoelasticity*, Chap. 2, pp. 16–17. Blaisdell, Massachusetts (1967).
8. E. H. Lee and T. G. Rogers, Solution of viscoelastic stress analysis problems using measured creep or relaxation functions. *J. Appl. Mech.* **33**, 127–133 (1963).
9. G. A. Costello and J. W. Phillips, Effective modulus of twisted wire cables. *J. Engng Mech. Div., Proc. American Soc. Civil Engrs*, **102**, EM1, 171–181 (1976).